Effect of Inclusion of Delayed First-Order Hold on the Stability of First-Order Sampled Data Proportional Control System

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Behavior of sampled data proportional control of a first-order system containing zero order hold is well documented (for example, Mosler et al. 1966; Luyben, 1972). The ultimate value of the proportional control constant is given by

$$K_u = \frac{1 + \exp(-T/\tau)}{1 - \exp(-T/\tau)} \tag{1}$$

The objective of this note is to develop a relationship for the ultimate gain when the zero order hold is replaced by a delayed first-order hold (abbreviated as DFOH). It is shown in this note that inclusion of such a holding device tolerates higher loop gain than the conventional zero-order hold.

DELAYED FIRST-ORDER HOLD

The output of a zero-order hold is discontinuous at sampling instants, while that of a DFOH is continuous. The output of DFOH are straight line segments which may be expressed as

$$Y(t) = X_{i-1} + (X_i - X_{i-1})\left(\frac{t - iT}{T}\right) \tag{2}$$

$$\text{for} \quad iT \leq t \leq (i+1)T$$

where X_j is the sampled input during the time jT and $(j+1)T$, and $y(t)$ is the continuous output. The DFOH builds straight line segments between the immediate past and the present values of input. A more detailed block diagram is given in Figure 1, where the relationship between the output $Y(s)$ and the input $X(s)$ can be obtained as

$$Y(s) = \left[\left(\frac{1 - e^{-Ts}}{s}\right)\left(\frac{(1/T)}{s}\right) - \left(\frac{1 - e^{-Ts}}{s}\right)e^{-Ts}\,\frac{(1/T)}{s}\right]X(s)$$

Hence, the transfer function for DFOH is

$$G_H(s) = \frac{Y(s)}{X(s)} = \left[\frac{(1 - e^{-Ts})^2}{Ts^2}\right] \tag{3}$$

The above relationship can also be obtained from the impulse response of the DFOH shown in Figure 1:

$$G_H(t) = \left(\frac{t}{T}\right)u(t) - \frac{2(t-T)}{T}u(t-T) + \frac{(t-2T)}{T}u(t-2T)$$

Taking Laplace transform of the above, we get

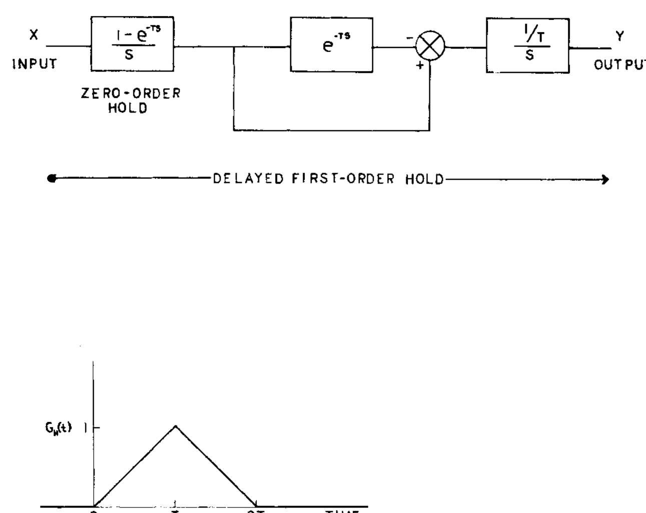

Fig. 1. Block diagram of delayed first-order hold. Impulse response of DFOH.

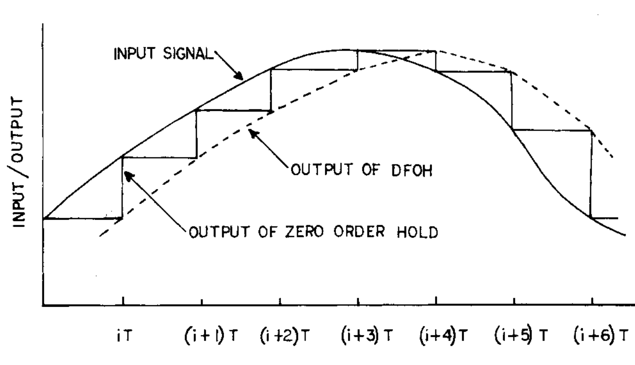

Fig. 2. Input-output relationship of a delayed first-order hold.

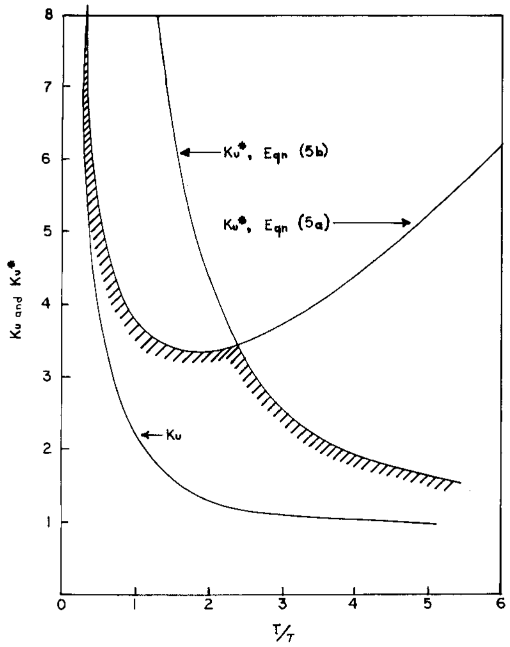

Fig. 3. Comparison of ultimate proportional loop gains for the two cases.

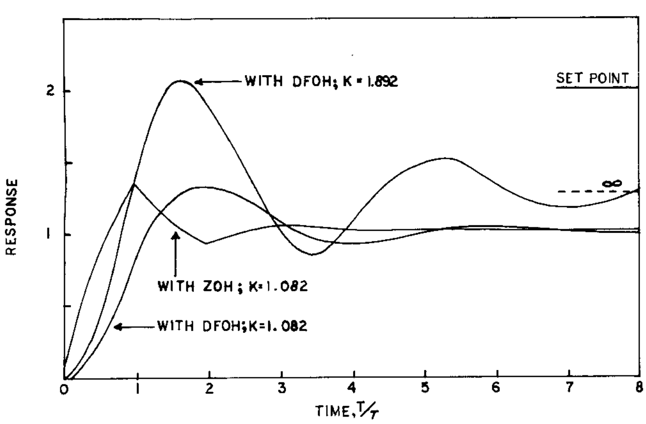

Fig. 4. Typical responses of a first-order system with zero-order hold and delayed first-order hold, $T = 1$.

$$G_H(s) = \frac{1}{T}\left[\frac{1}{s^2} - \frac{2e^{-2Ts}}{s^2} + \frac{e^{-2Ts}}{s^2}\right]$$

which can be factorized to the form of Equation (3) to give the transfer function of the DFOH.

Physical realization of DFOH in a computer control system requires the difference between the present and past input to be sent to an integrator of gain $1/T$ or the value $-(X_i - X_{i-1})/T$ to the input of an output integrator. The minus sign is incorporated because an integrator inverts the sign. Figure 2 shows a typical response of a DFOH.

STABILITY ANALYSIS

The characteristic equation in the Z domain of the sampled data proportional control system with the DFOH may be written as

$$z^2 + z\left[K - K\left(\frac{\tau}{T}\right)(1-b) - b\right] + K\left[\frac{\tau}{T}(1-b) - b\right] = 0 \quad (4)$$

where b is $\exp(-T/\tau)$. Use of bilinear transformation followed by the application of Routh's criterion yields the following constraints on the proportional gain, which guarantee stability:

$$K_u^* < \frac{1}{\left(\frac{\tau}{T}\right)(1-b) - b} \quad (5a)$$

$$K_u^* < \frac{1+b}{(1+b) - 2\frac{\tau}{T}(1-b)} \quad (5b)$$

The inequality in Equation (5*a*) is a stronger constraint for lower values of T/τ, while the inequality in Equation (5*b*) is the limiting constraint at higher values of T/τ. In any case, as shown in Figure 3, the ultimate value of the proportional constant is higher than in the case of zero-order hold.

TRANSIENT RESPONSE

Figure 4 shows typical responses of a first-order system with zero-order hold and DFOH to a step change in set point. With the proportional gain set to half of the ultimate gain K_u, the response with DFOH shows a slightly lower overshoot. In the same figure with the gain set to half of the ultimate gain K_u^*, the response shows a larger overshoot with faster speed of response and lower decay ratio.

DISCUSSION

For continuous control systems, inclusion of additional delay or lag decreases the ultimate proportional gain. In the present case of sampled data control, such inclusions actually enhance the stability of the system. This feature enables the use of higher loop gain.

ACKNOWLEDGMENT

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NOTATION

- b = $\exp(-T/\tau)$
- $G_H(s)$ = transfer function of DFOH
- $G_H(t)$ = impulse response of delayed first-order hold
- K_u = ultimate proportional gain or ultimate loop gain with zero-order hold
- K_u^* = ultimate proportional gain or ultimate loop gain with delayed first-order hold
- s = Laplace transform variable
- t = time
- T = sampling period
- $u(t)$ = unit step function
- τ = process time constant

LITERATURE CITED

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